Episode 3
MATHEMATICS WITHOUT CALCULATIONS

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1 Mathematical Thinking

Most people with a bachelor's degree when tasked to solve questions such as presented below would associate this problem with the discipline of mathematics.

> Zeno bought three kilos of pineapple at fifty rupees a kilo, and five cartons of milk at sixty rupees a carton. He hands a five hundred rupee note to the shop keeper. How much money should he get back?

Such a problem would test the ability to model real life situations expressed in the words and sentences of a natural language like English and transform this information to a mathematical model to come up with an answer.

This involves three steps:

1. **Abstraction**: taking information from the real world like three kilos, five cartons and sixty rupees, and transforming this into data that we can input into the world of mathematics. In this particular case, the move involves using the numbers 3, 50, 5, and 60 as inputs.
2. **Calculation:** using these inputs to arrive at the output, thus solving the problem. This step evaluates the ability to model real life situations using calculations that call for *multiplication* (3 x 50, 5 x 60), *addition* (150 + 300), and *subtraction* (500 – 450).

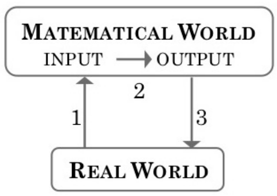

Figure 1

3. **Stating the solution:** requires return to the real world and stating the solution in terms of words and sentences. In this example the answer would be stated in English as fifty rupees.

Some individuals confronted with word problems such as the above example are fearful of making mistakes when doing the arithmetic calculations required to get a solution. For some reason they fear or even hate having to make even the simplest of these computations and therefore, shy away from solving such problems. This in some ways cripples them in their educational experience and in real world situations as well. Perhaps traumatic experiences learning math during their primary education gave them this fear or dislike.

For those students who fear math calculations and other who are not specialising in math or in a subject that requires applied math (e.g., physics, engineering, or economics), what is valuable in the study of math are the forms of thinking that it lends itself to.

Besides mathematical ***calculations***, the concepts of mathematical ***thinking*** and mathematical ***reasoning*** are important for processing information, no matter what the subject matter or situation may be. Young learners need to gain a sense of how this way of thinking is critical for their personal, public, and professional lives even after graduation. For those who are specialising in math and know the immense value of mathematical computations, they must also realize that there is more to math than mere calculation. Calculation after all, is only one of the components of thinking, the others being imagination, insight, and intuition.

The questions we seek to explore in this article are:

1. What is *mathematical thinking*, as distinct from calculation?
2. How do we design a course in *thinking like mathematicians*?
3. Can math be taught in a way that it develops the ability to use mathematical thinking in all aspects of education and life?
4. Would it make sense to introduce such a course as a compulsory foundation course for all bachelor's students (regardless of their subject major) in a four-year undergraduate program?

The answers to these questions lead to an even more fundamental question:

5. *What is the distinctive role of the knowledge system of mathematics in contributing to the other knowledge systems* present in the curricula for undergraduate programs?

This question relates directly to the title of this series, "*Designing and Implementing Curricula for Higher Education,*" and to the first article in the series, "Knowledge and Knowledge Systems."

2 Calculating

Calculating is a *mechanical process of reasoning* that requires an input and yields a specific output (typically numerical), regardless of who or what the calculating agent is (whether a human or a machine); the output is always the same.

In arithmetic, calculation is a mechanical process that takes a complex numerical representation as the input to yield another representation as an output. For instance:

Calculation	Arithmetic Operation
To calculate: ((20 x (50)) – (108 + 18))/5	
Input: 20 x 50 = 100	multiplication
108 + 18 = 126	addition
100 – 126 = –26	subtraction
–26 ÷ 5 = –5.2	division
Output: –5.2	

Each of the arithmetic operations in this example is a general mechanical procedure that follows specific rules. Addition, subtraction, multiplication, and division are representative examples of these types of operations.

Typically, math taught to children is geared towards mastering the skills of knowing and applying the rules (symbol manipulation), with little emphasis on an understanding of the concepts behind the rules. And these skills are what students are evaluated on.

For instance, only rarely do students raise the question why the product of two negative numbers is a positive number, while the sum of two negative numbers remains a negative number.

For instance, to calculate the product of multiplying 35 by 9, we use the following steps:

Step A	35 x 9	Right-align the numerals vertically.
Step B	5 x 9 = 45 3 x 9 = 27	Using multiplication tables, multiply the top numeral, one digit at a time, from the right, by each digit of the bottom numeral, starting from the right.
Step C	35 x 9 45 [9x5] + 27 [9x3] 315	Align the outputs of Step 2 as shown and add them to get the final output.

What the steps show is that: **35 x 9 = 315.**

Central to the understanding of such procedures is the understanding of the formal system of equations that use the equality symbol '='. An ***equation*** is a formal expression of the form:

X = Y

The variables X and Y represent the structure of an entity, (e.g., 35x9), and the symbol '=' says that the numerical value of X is the same as the numerical value of Y. For instance, the equation "(5x (10 – 3)) = 35" says that the numerical value on both sides of the equality symbol is 35.

Equations are governed by principles such as the following:

Transitivity:	If:	X = Y
	and:	Y = Z
	then:	X = Z
Transposition:	If:	X = Y
	then:	Y = X

Such principles are supplemented by procedural rules for operations, e.g.,

Substitution:	If the input has:	X = Y
	and:	Y = Z
	then:	in X = Y, replace the Y with Z.

That should give a sense of what we mean by arithmetic calculations using the formal language of equations.

3. Reasoning

Reasoning always has a set of premises, and a legitimate conclusion that follows from them. Calculation is a specific form of mechanical formal numerical reasoning, though not all forms of reasoning involve numbers or mathematics.

Consider the following example of what is called a *syllogism*:

Premise 1: Butterflies are insects.
Premise 2: Zeno was a butterfly.
Conclusion: Therefore, Zeno was an insect.

Syllogism is a special case of *reasoning with categories*. The syllogistic model of reasoning is composed of two premises and a conclusion:

~ a *major premise* that expresses a *subcategory relation* (e.g., Butterflies are a subcategory of insects);
~ a *minor premise* that expresses a *category membership relation* (e.g., Zeno was a member of the category of butterflies);
~ a *conclusion* derived from these two premises (e.g., Zeno was a member of the category of insects.)

In what we may call ***extended syllogism***, there can be multiple subcategory statements:

Premise 1: Animals are multicellular.
Premise 2: Vertebrates are animals.
Premise 3: Mammals are vertebrates.
Premise 4: Zeno was a mammal.
Conclusion: Therefore, Zeno was multicellular.

To turn to another form of reasoning, consider the following example of the model of ***implicational reasoning***, that employs premises of the form if *X (is true), then Y (is true)*:

Premise 1: If Zeno was a butterfly, then Aristotle loved ice-cream.
Premise 2: Zeno was a butterfly.
Conclusion: Therefore, Aristotle loved ice-cream.

To take another example, consider ***causal reasoning*** that employs premises of the form: X causes Y.

Reasoning from Cause to Effect

Causal Generalisation: Fire causes smoke.
A particular Observation: We see fire on that hill now.
Conclusion: Therefore, there is smoke on that hill now

Unlike (extended) syllogistic reasoning and implicational reasoning, causal reasoning allows us to make inferences in both directions: from cause to effect and from effect to cause. Here is an example:

Reasoning from Effect to Cause

Causal Generalisation: Fire causes smoke.
A particular Observation: We see smoke on that hill now.
Conclusion: Therefore, unless there is an alternative cause, we conclude that there is fire on that hill now.

The purpose of the discussion above section was to give readers a feel for the concept of reasoning and the various forms it can take. Although reasoning is central to all forms of academic knowledge, inquiry, and critical thinking, our aim at this time was not to help them develop their capacity to engage in reasoning.

4. Constructing Mathematical Theories

4.1 Reasoning from Definitions

We chose the title of this article, "Mathematics without Calculations," to emphasise the idea that at the heart of mathematical thinking are definitions, axioms, proofs, and theorems, and that calculation becomes relevant only after the conceptual structure of the theory has been expressed as a formal sructure.

The idea of mathematics without calculation would shock not only school students, but those specialising in mathematics in higher education programs, and perhaps even many who identify themselves as mathematicians. It is designed to provide students with the tools to develop mathematical thinking as part of higher order cognition (HOC); this idea provides a glimpse into a different vision of higher education

Here is an activity that illustrates that vision:

> *Define triangles to prove that straight-angled triangles* ***do not*** *exist; then define triangles to prove that straight-angled triangles* ***do*** *exist.*

This is a task we have tried out successfully with eighth grade students, as well as Year-1 undergraduate students. It is important to bear in mind that at the very outset of engaging with the task, we must define the concepts of *angle, straight angle, triangle,* and *straight-angled triangle.*

Here is a possible definition of the concept of angle:

> ANGLE (DEF): *Angle* is the magnitude of rotation at a vertex.
>
> VERTEX (DEF): A *Vertex* is a joint between two lines.

Suppose we have two lines AB and BC joined together at C as in Fig. 2:

Imagine line CA rotating around C such that it lies on top of CB. The magnitude of that rotation is angle ACB.

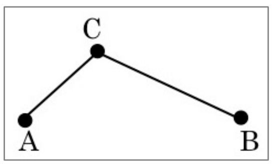

Figure 2

If line CA rotates around C until it returns to its initial position, we get a ***full rotation***. We can now define a right angle and a straight straight as follows:

> RIGHT ANGLE (**DEF**): A *right angle* is one quarter of a full rotation.
>
> STRAIGHT ANGLE (**DEF**): A *straight angle* is half of a full rotation.

This means that a straight angle is the sum of two right angles.

To say whether or not straight-angled triangles exist, we also need to define the concept of 'triangle'. Students would likely define it as:

> TRIANGLE (**DEF**): A *triangle* is a geometric object with three vertices and three straight line segments.

If they do come up with this definition, we ask them if Fig. 3 would constitute a triangle.

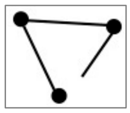

Figure 3

Fig. 3 has three vertices and three straight line segments. But it does not constitute a triangle. Hence, we need to revise the definition.

> TRIANGLE (**DEF: REV 1**) A *triangle* is a shape with three vertices, and three straight line segments **connecting them**.

We now return to our question: Do straight-angled triangles exist?

Take three straight lines AC, CB, and CA, configured as in Fig. 4:

Figure 4

Finally, what is a straight-angled triangle?

> STRAIGHT-ANGLED TRIANGLE **(DEF) A** *straight-angled triangle* is a triangle in which one of the angles is a straight angle.

Given this definition, it follows that Fig. 4 is indeed a straight-angled triangle. It has three straight lines (AC, CB, and BA) connecting three vertices (A, B, and C), where angle ACB is a straight angle.

Now, most students may have the intuition that ACB is not a triangle, probably because two of its angles have zero magnitude. But our definition of triangles does not say anything about not allowing zero angles, thereby disallowing straight angles. To align their definition with their judgment, they need to revise their definition of a triangle:

> TRIANGLE (**DEF: REV 2**) A *triangle* is a shape with three **non-colinear** vertices, and three straight line segments connecting them.

In a straight-angled triangle, the three vertices are colinear. So, our revised definition of triangles forbids the existence of such triangles. Hence straight-angled triangles do not exist.

What we have demonstrated here is the logical contradiction between the definition of triangles, as in DEF-REV 1, and the judgement that the representation in Fig. 4 is not a triangle. Given the principle that logical contradictions are prohibited in a body of knowledge (including in a theory), it follows that we must revise either the definition or the judgement.

It is important to note that the knowledge system of mathematics does not tell us whether it is the definition or the judgement, or both, that must be revised.

4.2 Reasoning from Axioms

A postulate is what we ***assume*** as a concept or a proposition of a theory. Both definitions and axioms are instances of postulates. Having seen an extended example of reasoning from definitions, it would be appropriate to look at an example of reasoning from axioms. For this, we will use a version of postulates associated with parallel lines, what are called parallel postulates.

Given two straight lines, our intuition — coming from what we have learnt from Euclidean geometry in our school textbooks — says that when extended on either side, they may either not intersect (as in Fig. 5a), or intersect on one of the sides (as in Fig. 5b), but not on both sides. The two lines are parallel if and only if they do not intersect.

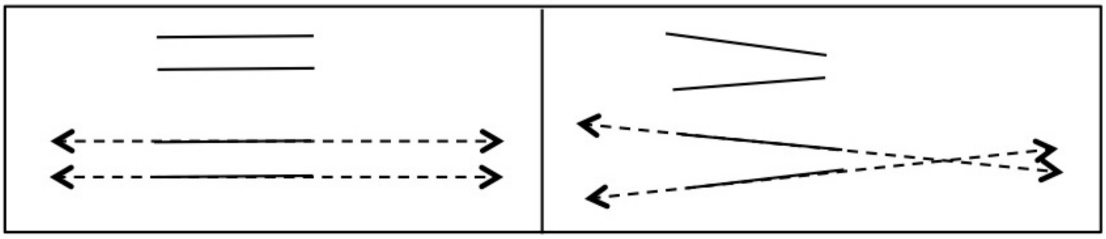

Figure 5a ***Figure 5b***

Let us state as an axiom our intuition that two straight lines cannot intersect on both sides:

> Axion A: No two straight lines, even when extended indefinitely, can intersect at two distinct points.

Contrary to what our intuition tells us, let us assume that they do intersect at two points, and state this as an alternative axiom:

> Axiom B: Any two straight lines when extended, intersect at two distinct points.

The choice of Axiom A results in a geometry of flat surfaces. The Euclidean Geometry presented in school textbooks is an example of Flat Surface Geometry (FSG). The choice of Axiom B results in a geometry of spherical surfaces. FSG and SSG yield distinct theorems. To take an example, consider the sum of angles in a triangle. In FSG, we can prove that the sum of angles in a triangle is two right angles. Anyone who has completed ten years of school education is familiar with this theorem under the name *Angle Sum Theorem*. In SSG, on the other hand, the sum of angles in a triangle is more than two right angles, up to three right angles.

Here is a way of understanding the SSG theorem. Imagine that you are travelling in a straight line on the surface of the earth from point A on the equator to point B at the North Pole, describing line AB. At the North pole, you change your direction, and travel in a straight line perpendicular to AB, until you are at the equator. Call it point C, forming a straight line BC. Now you change your direction again, and travel along the equator to point A. Line CA will be perpendicular to line BC. Each of the angles ABC, BCA, and CAB in triangle ABC will be a right angle, making the sum three right angles.

If you find it difficult to imagine what is described above, try this. Take a sweet lime (moosambi) and cut it through the center. You now have two hemispheres, with a flat cross section. Take one of the hemispheres and cut it in such a way that the new cross section is perpendicular to the earlier one.

You now have two pieces. Cut one of them, such that the new cross section is perpendicular to both the previous ones. If you examine one of the pieces, you will see a triangular curved surface, where each of the angles is a right angle. So, the sum of angles of that triangle will be three right angles.

We are not going to prove the conjectures in FSG or SSG, to establish them as theorems in their respective geometries, but you should now be able to understand the conjectures.

4.3 Reasoning from Postulates

In Section 7 of our article, 'From Experience to Knowledge,' (Episode 2 in this series in KSHEC's *Higher Education Matters*, Volume 1, Issue #7), we said that a theory is composed of:

A set of postulates (definitions, axioms/laws).
A set of conclusions (called theorems in mathematics, and predictions in science); and
A set of derivations of conclusions from postulates (called *proofs* in mathematics).

The form of mathematical reasoning that derives conclusions from the postulates of the theory is called ***classical deductive reasoning***. This includes reasoning with categories and implicational reasoning, among others.

We now turn to the challenge of helping undergraduate students to construct mathematical theories without necessarily using numerical calculations, but providing adequate practice in reasoning to derive theorems.

4.4 Matchstick Geometry

Going back to mathematical thinking without mathematical calculation, we need to have an understanding of theory construction in pure mathematics, and of the use of that theory to model phenomena in applied mathematics (including science, engineering, and technology). In this article, we only deal with pure mathematics.

To exemplify theory construction, we will use what is called Matchstick Geometry (MG). This form of geometry is ideal for helping students to get an intuitive feel for what it is like to construct a mathematical theory, and to give them a glimpse into what is distinctive about the knowledge system of mathematics.

Consider the representation of a matchstick in Fig. 6.


Figure 6

In this figure:

the stick of the matchstick is represented by the straight line, which represents a ***straight line segment*** in the geometry; and
the coloured tip of the matchstick is represented by the black dot, which represents what we call a ***point*** in geometry.

Let us ask a question about such line segments with a point at one end:

Q1: Is it possible to construct an equilateral triangle with such geometric straight line segments?

If we consider the physical world we live in, with matchboxes and matchsticks, the answer is obviously yes, and the result of that configuration of matchsticks would look like Fig. 7:

Figure 7

Fig. 7 is how we would represent the abstract geometric figure we call 'equilateral triangle'.

Let us take another question:

Q2: Is it possible to put together equilateral triangles to construct a regular hexagon with such geometric straight line segments?

To answer this question, eighth grade students may put together six equilateral triangles as in Fig. 8,

and then remove the radius sticks such that the result is as in Fig. 9.

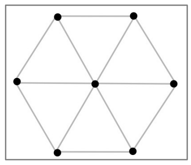

Figure 8

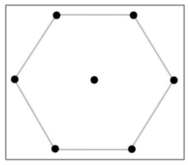

Figure 9

We must acknowledge that what we have sketched above is only the beginning of an answer to the question we raised. To answer the question satisfactorily, we need to understand the definitions of equilateral triangles, polygons, regular polygons, and hexagons within MG. Those whose education in mathematics is restricted to Euclidean geometry would require considerable assistance in meeting that challenge.

Yet another question is:

Q3: The distance from the center of the hexagon to the vertices has the length of a matchstick.
Can this hexagon be considered a circle in MG?

Most readers are likely to say that such a hexagon is not a circle. But the question is not whether it *looks* to us like a circle, but whether the conjecture (or its contradiction) follows logically from the postulates of the theory.

To do that, we need to set up postulates on this abstract world. Suppose we begin with the following postulates:

P-1: Points have zero magnitude. [In a two-dimensional geometry, this means that points have no length or breadth, following Euclid.]

P-2. Lines have length, but no breadth. [Faithful to Euclid.]

Now comes the departure from Euclidean Geometry:

P-3: The distance between adjacent points on a dimension is fixed.

P-4: The line between two adjacent points is an atomic line.

P-5: A composite line is composed of atomic lines, such that when two atomic lines join, there is only one point at the joint.

P1-P5 constitute the heart of Matchstick Geometry. Think of matchsticks as atomic lines that can be combined to form composite lines (P-5). The length of the matchstick is the length of the atomic line (P-4), and hence, in MG, all atomic lines have the same length. We also need to add:

A. An atomic line has only one point (at one of the two ends).

B. A circle (i.e., a circular finite line) is a composite line.

C. The length of a composite line can be defined as the sum of the atomic lines it is composed of.

Given the above, we can ask:

1. Is a circle a regular polygon?
   Prove that the answer is YES in MG, and NO in Euclidean geometry.
2. A square is a rectangle with equal sides. Do squares exist in MG?
   Prove your answer, whether yes or no.
3. In Euclidean geometry, any line can be bisected. What about in MG?
   Prove your answer.
4. Do the following shapes exist in MG? Propose and prove conjectures:
   a. Right-Angled Triangles.
   b. Straight-Angled Triangles
   c. Quadrilaterals that are not squares.
5. Can every shape that exists in Euclidean geometry exist in MG?
   Prove your answer.

Now, answering (4a) would require a bit of help:

There are indeed right angled triangles in MG, in which the hypotenuse (the side opposite the right angle) of a right angled triangle follows the theorem that says that the square of the hypotenuse is equal to the sum of the squares of the sides adjacent to the right angle. For instance, if the hypotenuse is 5 (whose square is 25), one side adjacent to the right angle is 4 (whose square is 16), and the other side is 3 (whose square is 9), so that we get 25 = 9 + 16.

Such a triangle is possible in MG. But can we make a right angle triangle whose sides adjacent to the right angle are four each, giving us an isosceles triangle? Try constructing such a triangle in MG.

Even though the previous sections have been talking mostly about reasoning, it is important to bear in mind that mathematical thinking is not just reasoning or formal logic. It crucially involves other mental capacities such as those of postulating definitions and axioms, abstracting, generalising, developing and recognising insight, and intuition. This is true even in the case of discovering proofs in pure mathematics and applying theorems in applied mathematics.

A word of caution. The theory of matchstick geometry developed in this section is that of a toy theory for explanatory purposes. It cannot do many things that a professional mathematician would want to do. Our intention is not to give a full-fledged theory, but simply to provide a feel for what it is like to construct a theory in mathematics.

5 Summary

What we have outlined in this article is *thinking* and *calculating* as two important aspects of mathematical inquiry. Of these, making numerical calculations using representations of numerals and the rules governing arithmetic operations on the numerals is what we are taught in school as mathematics. The journey that we have undertaken in this article is to walk with the reader through other part of mathematics which we believe to be the core of mathematical thinking in pure mathematics. We hope that the journey has given you a feel for what the knowledge system of mathematics shares with other systems of academic knowledge, and what distinguishes it from them.

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