DESIGNING AND IMPLEMENTING CURRICULA FOR HIGHER EDUCATION

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EPISODE 4

UNDERSTANDING AND EVALUATING PROOFS

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1 Introduction

1.1 Proof

In mathematics, the term ***proof*** refers to a rational argument that convinces mathematicians that a statement that is advanced as a claim should be accepted as true. We may say that a proof is a response to the open minded rational sceptic's question "Why should I accept that claim?"

When a course or a textbook in mathematics asserts that the sum of angles in a triangle is 180°, we expect a rational open minded sceptic (ROMS) to ask: "Why should I accept that claim as true?" We will examine possible responses to such questions, and the nature of such proofs, later in this article.

Pure mathematics distinguishes between conjectures and theorems. A ***conjecture*** is a statement that a knowledge seeker feels or senses to be true. Its counterpart in science is called a ***hypothesis***. For a conjecture to be accepted as true in the community of mathematicians, the defender of the conjecture has to support it with a proof. And once the proof is accepted as valid by the community, it becomes a ***theorem***.

We would like to suggest that the norm of supporting assertions with proofs be extended to all academic disciplines. In this article, we will study the nature of mathematical proofs to find out how some of its desirable properties can be incorporated into proofs in other disciplines.

1.2 Premises, Derivation, and Conclusion

An important feature of proofs in mathematics is the explicitness of the premises and conclusion, as well as the steps that derive the conclusion from the premises. What do we mean by that?

Suppose two people, A and B, are looking at a creature in front of them. Here is a conversation between them.

A. That creature has compound eyes.

B. Why should I believe that it has compound eyes? I can't even see its eyes, let alone check if it does have compound eyes.

A. I don't need to actually see its eyes to conclude that it has eyes and that they are compound eyes. You can see its legs, right? How many legs does it have?

B. Six.

A. Well, so let me give you a proof to convince you that it has compound eyes. **P** stands for *premise*, and **C** for *conclusion*.

P 1: This organism has six legs.
P 2: All organisms with six legs have compound eyes.
C: From Premises 1 and 2, it follows logically that this organism has compound eyes.

B. Wait, how does the conclusion follow from the premises?

A. Oh, okay. It follows from a rule of inference in what Aristotle called syllogistic logic. The rule is:

Given premises of the form:
P-1 All X is Y. (e.g. *All humans are mortal.*
P-2 z is X. *z is a human.*
it is legitimate to conclude that
C: z is Y. *z is mortal.*)

This is equivalent to:
P-1 All X has property Y.
P-2 z is X.
it is legitimate to conclude that
C: z has property Y.

If you accept this rule of inference, and agree that the proof above has the structure: "If P-1 and P-2, then C," then you are forced to accept the conclusion: "This organism has compound eyes."

Let us take a parallel scenario, this time from geometry:

A: The sum of angles in that geometric object is 180 degrees.

B: Why should I accept that assertion?

A: Well, let me give you a proof to convince you.

P 1: That geometric object is a triangle.
P 2: The sum of angles in a triangle is 180 degrees.
C : From Premises 1-2, it follows logically that the sum of angles in that geometric object is 180 degrees.

Statements about all members of a category are called ***major premises***, while those about a particular member or a sample of members is called a ***minor premise***. In our examples, the first premise is a minor premise and the second premise is a major premise.

Mathematical proofs are not primarily used to prove judgements such as "That object is a triangle," or "That object is not a right-angled triangle." They are used for general statements like: “Right-angled triangles are not equilateral.”

In scientific inquiry, statements of minor premises such as “x is a male human,” and “x is i centimeters tall,” “y is a female human,”, and “y is j centimeters tall,” are called *observational reports* (also called *data points*.) Based on a sample of such observational resports, we prove ***observational generalisations***, for instance, “In the human species, males are taller than females.”

But given statements such as: “x is a member of the human species,” and “x is male,” how do we prove (or disprove) the statements? Such questions about category membership involve definitions of the categories, and are important for all branches of academic inquiry.

The terminology of minor and major premises comes from the system of logic called *syllogism* developed by the ancient Greek philosopher, Aristotle. This article takes the position that it is important to doubt and question not only the conclusion, but also the minor premises, the major premises, and the steps in the derivation of the conclusion. As our subject matter, we take statements about triangles in geometry.

1.3 Explicitness of Postulates

When someone doubts and questions the credibility of premises in a proof, we provide proofs for those premises. But the premises in those proofs are also subject to doubting and questioning. That process could go on indefinitely, ad infinitum. But that would be unproductive. So we stop at some stage, with the decision that we will not delve further.

Drawing on the term *postulate*, found in discussions of Euclidean geometry, we will modify the term and use it to refer to those premises that we cannot prove. Postulates are either definitions or axioms. Mathematical proofs are governed by the norm that definitions and axioms must be explicitly articulated.

Why should we adopt this norm? Because our knowledge is fallible: what we believe to be true may turn out to be false, and hence we must try our best to minimise falsehood and maximise truth. Explicit articulation of the premises, including those that we cannot rationally defend, is a way of identifying untenable assumptions that we may have taken for granted and may not have noticed otherwise.

1.4 Algorithmic Proofs

An algorithm is a set of mechanical procedures that take an input and yield an output. It is a set of unambiguous instructions, a recipe that even a mindless robot can execute. An example that would be familiar to anyone who has completed primary school education is that of the arithmetic operations of addition, subtraction, multiplication, and division on numbers represented with multiple digits. If you can find the product of, say, 859 and 28, you are using an algorithm for calculating the answer.

In the course of the evolution of proofs from antiquity to the modern times, mathematical proofs have become more and more algorithmic. Suppose you make the claim that the product of 859 and 428 is 367652, and someone asks: “Why should I accept that claim as true?” Your response would be to apply the procedure of multiple digit multiplication, and derive 367652 from 859x428, to prove that your assertion is true.

In this article, we try to fruitfully extend the concept of proofs to all domains of academic knowledge.

2 Is that a Triangle?

Let us look at the concept of triangles as a simple example. We will examine the possible definitions and premises that lead us to prove that a given object is or is not a triangle.

Consider the following figures:

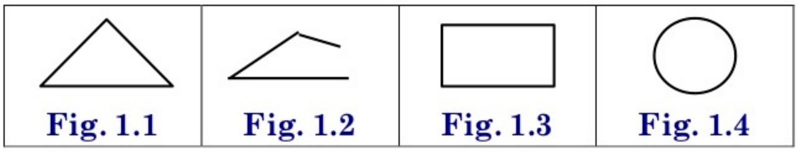


Fig. 1.1 **Fig. 1.2** **Fig. 1.3** **Fig. 1.4**

FIGURE 1

Children familiar with the words *triangle*, *rectangle*, and *circle* would judge Fig. 1.1 to be a triangle, Fig. 1.3 to be a rectangle, and Fig. 1.4 to be a circle. They would also claim that Fig. 1.2 is not a triangle. In order to *prove* these claims, however, they would have to learn the art and craft of constructing proofs.

To prove the claim that Fig. 1.1 is a triangle and the remaining figures are not triangles, we need a definition of triangles. Suppose we postulate the following definition:

> **DEF 1** (TRIANGLE): A triangle is a closed shape with exactly three angles.

Given this definition, we can offer the following proofs for the claims below:

CLAIM 1: Fig. 1.1 is a triangle.
Proof 1
P 1: A triangle is a closed shape with exactly three angles. (DEF.1)
P 2: Fig. 1.1 is a closed shape.
P 3: Fig. 1.1. has exactly three angles.
C: Therefore, Fig. 1.1 is a triangle.

CLAIM 2: Fig. 1.2 is not a triangle.
Proof 2
P 1: A triangle is a closed shape with exactly three angles. (DEF.1)
P 2: Fig. 1.2 is not a closed shape.
C: Therefore, Fig. 1.2 is not a triangle.

CLAIM 3: Fig. 1.3 is not a triangle.

Proof 3

P 1: A triangle is a closed shape with exactly three angles. (DEF.1)
P 2: Fig. 1.3 is a closed shape, but it has four angles.
C: Therefore, Fig. 1.3 is not a triangle.

CLAIM 4: Fig. 1.4 is not a triangle.

Proof 4

P 1: A triangle is a closed shape with exactly three angles. (DEF.1)
P 2: Fig. 1.4 is a closed shape, but it has no angles.
C: Therefore, Fig. 1.4 is not a triangle.

Definition 1 chooses the number of angles as a definitional characteristic of triangles. Suppose, instead, we choose the number of straight lines as the definitional characteristic, to propose definition 2:

DEF 2 (TRIANGLE): A triangle is a closed shape with exactly three straight lines.

If we choose to go by this definition, the proofs will need to be different. (We omit the proof for claim 2 because it holds with DEF 2 as well.)

CLAIM 5: Fig. 1.1 is a triangle. [same as Claim 1]

Proof 5

P 1: A triangle is a closed shape with exactly three stright lines. (DEF.2)
P 2: Fig. 1.1 is a closed shape.
P 3: Fig. 1.1. has exactly three stright lines.
C: Therefore, Fig. 1.1 is a triangle.

CLAIM 6: Fig. 1.3 is not a triangle. [same as Claim 3]

Proof 6

P 1: A triangle is a closed shape with exactly three stright lines. (DEF.2)
P 2: Fig. 1.3 is a closed shape, but it has four stright lines.
C: Therefore, Fig. 1.3 is not a triangle.

CLAIM 7: Fig. 1.4 is not a triangle.

Proof 7

P 1: A triangle is a closed shape with exactly three stright lines. (DEF.2)
P 2: Fig. 1.4 is a closed shape, but it has no straight lines.
C: Therefore, Fig. 1.4 is not a triangle.

While either of these definitions is sufficient to provide rational justification for the judgements on Figs. 1.1 to 1.4, they are not sufficient to provide rational justification for our judgements on Figs. 2.1 to 2.3 below.

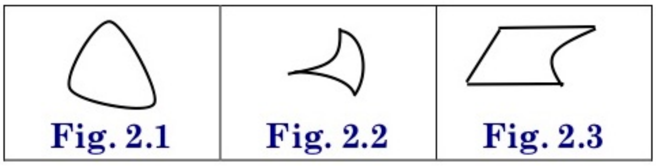


Fig. 2.1 Fig. 2.2 Fig. 2.3

FIGURE 2

Definition 1 does not tell us why we judge 2.2 to be ‘not a triangle’, as the figure satisfies the requirement of three angles. Definition 2 does not tell us why we judge 2.1 and 2.3 to be ‘not triangles’, as the figures satisfy the requirement of having exactly three straight lines.

We will leave it to the readers to engage with the challenge of constructing a definition of triangles such that it correctly predicts the distinction between Fig. 1.1 on the one hand, and Figs. 1.2 to 1.4 and 2.1 to 2.3 on the other. By going through simple examples such as this one, we can gain confidence and build up our mental muscles to construct more complex theories and proofs, not only within but also outside geometry, and outside mathematics itself.

3. Is that an Equilateral/Right-angled Triangle

3.1 Subcategories of Triangles

Learners who are familiar with the terminology of equilateral and right-angled triangles would judge Fig. 3.1 to be an equilateral triangle, and Fig. 3.2 to be a right-angled triangle:

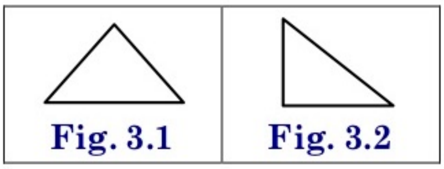


Fig. 3.1 **Fig. 3.2**

FIGURE 3

What is the proof in support of the judgements that 3.1 is an equilateral triangle and 3.2 is a right-angled triangle?

3.2 Equilateral Triangles

An equilateral triangle is one in which all sides are equal. How do we know that all sides are equal in the case of Fig. 3.1? We cannot measure the length with a ruler, because, to make a measurement, we need to see the line. Since lines in Euclidean geometry have zero breadth, they are invisible. So Fig. 3.1 is not a triangle, it is a representation of a triangle, in which lines do have breadth.

Euclid’s solution to the above problem is to replace ‘equal’ with ‘congruent’. If in our mind’s eye, we place one of the lines on top of another, and the two coincide exactly, they are congruent. It does not matter what their lengths are in terms of millimetres, centimeters, kilometres, or light years.

3.3 Right-angled Triangles

Precisely the same problem appears in the case of Fig. 3.2. The textbook definition of ‘right angle’ is that when measured with a protractor, it is 90° . But we cannot measure the angle between two lines with a protractor, as lines are invisible in Euclidean geometry. Hence, we need to look for a definition that does not involve measurement.

That alternative would be in terms of rotation. Here is how we go ahead with the mental construction of straight angles and right angles.

> In our mind’s eye, we take a line segment AB and rotate it around A, such that B revolves around A, and returns to its original location.
>
> That would be a ***full rotation*** of the line segment.
>
> Now we say that:
>
> Half a full rotation is a ***straight angle.***
>
> A quarter of a full rotation is a ***right angle.***

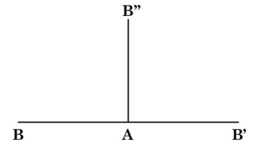


FIGURE 4

After half a rotation, if point B is at location B’, and B, A, and B’ are co-linear (i.e., on the same straight line), then angle BAB’ is a straight angle. If point B is at B” after a quarter rotation, such that angles BAB” and B’AB” are congruent, then they are both right angles.

We view triangles, equilateral triangles, and right-angled triangles as geometric objects. Now, Euclid’s geometry also contains procedures for the construction of geometric objects. Examples include procedures to construct circles of given radii, bisect a straight line segment, bisect an angl, construct a line perpendicular to another line at a given point, and so on. Notice that in our definition of straight angle and right angle, we appealed to the procedure of rotation.

Notice also that at the heart of this section is the importance of definitions in mathematical proofs.

4 Straight-angled Triangles

4.1 What is a Straight-angled Triangle?

Suppose someone asks us to draw a straight-angled triangle. Most of us would say that it is impossible to draw one. If so, our interlocutor may ask: how come we can draw equilateral triangles, right-angled triangles, but not straight-angled triangles? Can we try to draw one?

Let us approach this question from a different perspective. Draw triangle ABC, with angle A at the top, and side BC as the base, as in Fig. 4.1. Now, move point A down towards BC as in Fig. 4.2. Continue moving it down, till it almost touches BC, as in Fig. 4.3. Now move it further such A is on BC, as in Fig. 4.4. Continue moving A down, as in Fig. 4.5. Are all the figures, Figs. 4.1 to 4.5, triangles?

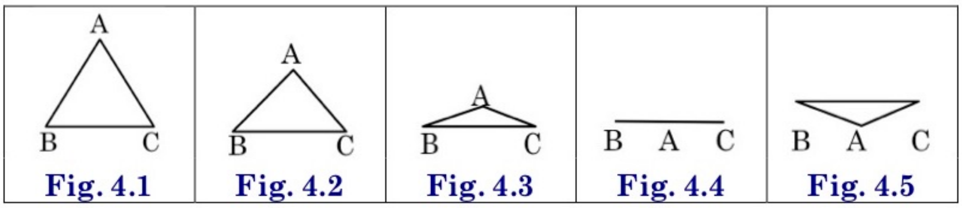


Fig. 4.1 **Fig. 4.2** **Fig. 4.3** **Fig. 4.4** **Fig. 4.5**

FIGURE 4

4.2 Do Straight-Angled Triangles Exist?

We are now ready to engage with the question:

> *Do straight-angled triangles exist?*

In Fig. 4.4, the vertices A, B, and C are connected through straight line segments BA, AC, and BC. If you judge Fig. 4.4 to be a triangle, then straight-angled triangles do exist. But if you judge it to be not a triangle, then straight-angled triangles do not exist. How do we make sense of this situation?

Let us ask: What would be the definition of triangles on the basis of which you can provide rational justification for each of those two opposing judgements? We leave that question open for you.

Here is a hint of how you would need to proceed: Come up with a definition of the concept of triangles such that Fig. 4.4 is a triangle. And then come up with another definition of triangles such that Fig. 4.4 is not a triangle.

Once you have a definition that leads to the conclusion that straight-angled triangles do exist, engage with the following questions:

- What is the area of a straight-angled triangle?
- Are there quadrilaterals, pentagons, and hexagons with all their vertices co-linear? If so, what would be their areas?

Having gone through these mental activities, you should now be ready to engage with the following question:

> What is the difference between the way we prove our answers to questions like: “Do straight-angled triangles exist?” and “Do polygons with zero area exist?” on the one hand, and “Do mermaids exist?”, “Do white crows exist?” and “Do electrons exist?” on the other.

5 The Angle Sum Theorem

5.1 Proving the Angle Sum Theorem

We began this article by examining the nature of proofs for minor premises in Aristotelian syllogism — claims about each of the entities. We then moved to a discussion of existence proofs to illustrate the importance of definitions in a proof. We will now turn to the challenge of doubting and questioning every step in the derivation of a proof with multiple steps. For this, we will use the familiar Angle Sum Theorem.

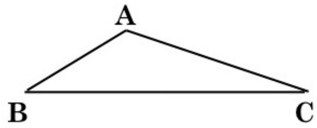
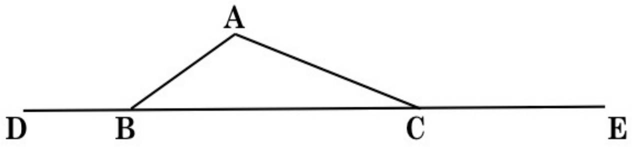
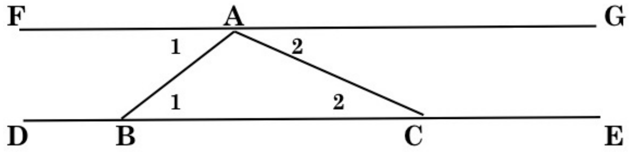
The theorem states:

> The sum of angles in a triangle is two right angles (180° if we view a full rotation as being composed of 360 units that we call degrees).

The proof of the angle sum theorem is usually given as follows:

> To prove: The sum of angles in a triangle is two right angles.

PROOF:

<table>
<tr><td>Step 1: Take any triangle ABC.</td><td>
<i>FIGURE 5</i></td></tr>
<tr><td>Step 2: Extend the line BC on both sides, to form a line of any length DBCE.</td><td>
<i>FIGURE 6</i></td></tr>
<tr><td>Step 3: Through A, draw a line FG parallel to DBCE.</td><td>
<i>FIGURE 7</i></td></tr>
<tr><td colspan="2">Step 4: Angle FAB and Angle ABC are alternate interior angles (angles of the Z shape).</td></tr>
<tr><td colspan="2">Step 5: By the alternative interior angles theorem of parallel lines, angle FAB and angle ABC are congruent. (marked as 1)</td></tr>
<tr><td colspan="2">Step 6: By the same reasoning as in step 5, angle GAC and angle ACB are congruent. (marked as 2)</td></tr>
<tr><td colspan="2">Step 7: Angle FAG (the sum of angles FAB, BAC, and CAG) is a straight angle (by definition of straight angles)</td></tr>
<tr><td colspan="2">Step 8: Given steps 5, 6 ,7 and 8, it follows that the sum of angles ABC, BAC, and CAB is a straight angle.</td></tr>
<tr><td colspan="2">Step 9: Hence, in any triangle, the sum of angles is a straight angle, that is, two right angles.
(QED)</td></tr>
</table>

[QED is an abbreviation for the Latin phrase *Quod Erat Demonstrandum*, meaning "that which was to be demonstrated," used at the end of proofs to signify completion.]

[For a slightly different proof of this theorem, watch the 5-minute YouTube video “Proof: Sum of measures of angles in a triangle are 180 | Geometry | Khan Academy” https://www.youtube.com/watch?v=6s1CI3uuhko&t=95s).]

This proof has 9 steps. We could question each of the steps, and ask: “How do we know that?” For instance, we could question step 3 by asking, “How do we know that through every point outside a straight line AB we can draw a straight line parallel to AB. The answer would be an axiom, called the Parallel Postulate in Euclidean geometry. If we question step 5, the answer would be a proof for this theorem. And questioning some of the steps in that proof would call for other proofs, until we get to the axioms and definitions that cannot be proved.

That, in essence, is how we critically evaluate the validity of multiple step proofs.

5.2 Refuting the Angle Sum Theorem

We have already proved the Angle Sum Theorem, and accepted the validity of that proof. So we must accept the theorem as true.

Now consider the following argument:

> Take any triangle ABC. Put an arbitrary point D on AB. Given the definition of straight angles, angle ADB is a straight angle (180 °). Since there are infinitely many points on every straight-line segment, however small, it follows that every triangle has infinitely many straight angles, and hence the sum of angles in a triangle is infinitely many, not just two right angles. Hence the Angle Sum Theorem is false.

This creates a problem for us. We have two proofs, both of which we accept as valid. Hence we must accept both conclusions. But the two conclusions are logically contradictory. How do we resolve this problem? .

Suppose we distinguish the concept of POINTS and of VERTICES, and define angles in terms of vertices — the junction between two lines. If so, a point on a straight line does not give rise to an angle, while a vertex on a line does. The refutation of the Angle Sum Theorem involved putting a point on a line, and not a vertex. Hence the conclusion that it results in a straight angle is not legitimate.

Regardless of the number of points on a line, a triangle has only three vertices, and therefore, only three angles. This rules out the possibility of infinitely many straight angles in a triangle.

The above activity is a reminder of the need to carefully scrutinise the validity of proofs in what we read or listen to, regardless of whether it is in textbooks, on the internet, or in social media.

6. Pythagoras' theorem

6.1 Two Versions of the Theorem

The Pythagoras' Theorem (PT) was originally stated in terms of areas:

> *PT version A:*
> If a triangle is right-angled,
> then the area of the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares of the other two sides,
> and vice versa.

In this formulation, there is no equality relation in terms of the numbers assigned to the variables of lengths of the sides.

Subsequently, however, PT was stated in terms of the numbers assigned to the lengths of the sides:

> *PT version B:*
> Given a triangle whose sides are of lengths a, b, and c, respectively,
> if the triangle is right-angled and c is the hypotenuse,,
> then $c^2 = a^2 + b^2$
> and vice versa.

Notice that while *PT version A* is about the areas of squares, where the concept denoted by the term ***square*** refers to a geometric object, *PT version B* is about a *number* assigned to a line segment multiplied by itself, where multiplication comes under number theory. While length is the amount of space occupied by a one-dimensional object, area is the amount of space occupied by a two-dimensional object.

A rational open-minded sceptic should doubt and question the legitimacy of the bridge between the concept of area in geometry and the concept of multiplication in number theory. Having planted that seed of doubt in the readers' minds, however, we will not investigate that issue in this short article: we leave it to the readers to engage with it.

Let us turn to the proofs of *PT version A* and *PT version B*.

There are a number of proofs for PT. We would suggest that the readers study at least a few of them, or at least Euclid's proof of *PT version B* and Bhaskara's proof of *PT version A*. These are discussed in the article, "Many Ways to QED The Pythagorean Theorem," which appeared in the magazine *At Right Angles*, Volume 1 No. 1, June 2012.

While going through these proofs, it may be useful to bear in mind that Euclid's proof is ***axiomatic*** in the following sense:

(a) It has a clear ***structure*** of Premises, steps of Derivation, and Conclusion.

(b) For each step in the derivation of the conjecture to be proved, a rational open-minded sceptic (ROMS) can ask: "How does step N logically follow from the preceding steps?" To this question, the defender of the claim (DC) would specify the theorems that form the implicit steps. In response, the ROMS may still ask for proofs for those theorems. At the end of such an exchange, the ultimate premises of the derivation would be the ***definitions*** and ***axioms*** of the theory, so this will not be an infinite regress.

(c) In a similar manner, if the ROMS examining one of the words/phrases in the sentences that express the propositions of the derivation asks: "What does that word/phrase mean?" the DC is expected to provide a definition. The ROMS may pick out one or more of the terms in the definitions, and ask for definitions of those terms. At the end of such an exchange, the DC would treat the remaining concepts denoted by the words/phrases as ***undefined*** or ***primitive*** concepts.

(d) Both axioms and primitive concepts are required to be ***as few as possible***. (Occam's Razor).

(e) The ***derivation*** of the conclusion from the premises is through classical deductive reasoning.

(f) The conclusion is the ***claim*** to be proved.

In contrast, Bhaskara's proof does not expect the reader to be a ROMS who insists on several rounds of rigorous questioning. Instead, it assumes that the readers' intuitive understanding of the words, phrases, and sentences is sufficient for them to understand the propositions of premises, derivation, and conclusion.

6.2 Why Ask for Definitions?

Why ask for definitions? Consider the following examples of 'proof' for the proposition that Xeno is an animal:

All mammals are vertebrates.
All vertebrates are animals.
Xeno is a mammal.
Therefore, Xeno is an animal."

Most readers do not ask which rules of inference are used in a proof, and which system of logic sanctions each step in the derivation. Nor do we ask for definitions of mammal, vertebrate, and animal. In the epistemic culture outside the 'academic' one, a ROMS described above would be perceived as nit-picking. Why should the terms expressed by the words, *animal, vertebrate* and *mammal* be defined, why do we need to provide the steps of derivation, and ensure that the derivation is valid?

These questions point to an important difference between the epistemic culture of proofs in Ancient Indian mathematics and Ancient Greek mathematics. As we see it, this difference is illustrated in the respective proofs of PT of Bhaskara and of Euclid. Understanding these differences in the epistemic cultures of proofs is central to our understanding of how knowledge systems vary across not only cultures but also across time.

7 Proofs in Knowledge Systems

In Episode 1 of this series, we discussed the concept of knowledge systems, and went through some of the special characteristics of the system of academic knowledge, transmitted through textbooks, encyclopedias, and research works. In Episode 2, we discussed different kinds of knowledge systems, including those of experiential knowledge, pre-academic knowledge, and also academic knowledge as the body of knowledge accepted as credible by the community of academics. And in Episode 3, we explored the knowledge system of mathematical inquiry,

An important aspect of the knowledge system of mathematical inquiry is that of proof. Having gone through a small sample of different kinds of proofs in geometry, we are now ready to understand the importance of questions like:

“What is a proof?” “Does the epistemic culture of a community require proofs for assertions?” and “What are the criteria for accepting an alleged proof as valid in a given community?” We might even evaluate those criteria themselves.

We began our journey with the claims, “X is a triangle,” and “X is an equilateral/right-angled triangle,” and explored proofs for such claims, which are statements about the particular (the minor premises in syllogistic reasoning). The X’s in these claims are particular members of a category of entities.

We then explored existence proofs for claims, illustrated by “Straight-angled triangles do not exist,” and “Straight-angled triangles do exist.” In statements of the form “X exists,” and “X does not exist,” X is a category of entities, so these proofs demand justification for the major premises in syllogisms. We also presented two versions of the Pythagorean theorem, examined different kinds of proofs of the theorem, demonstrated how proofs can vary depending on epistemic cultures.

In the course of our journey, we noticed that the critical scrutiny of proofs often sheds light on steps in the derivation which do not follow logically from the previous steps. In some cases, such gaps lead to alternative definitions and alternative axioms that can result in theorems in one theory being logically inconsistent with theorems in another. Even the so-called ‘Euclidean theory of geometry” is a family of many such logically incompatible theories.

That different epistemic cultures of proofs exist even within the study of triangles in geometry points to the importance of investigating both the ***diversity of knowledge systems*** and the ***evolution of that diversity*** from the ‘ancient’ to the ‘modern’. Axiomatic proofs that derive conclusions from definitions and axioms are central to pure mathematics, and theoretical laws to scientific theories in the physical and cognitive sciences (though this is not true in many other pursuits). Whether such a culture of proofs needs to be adopted in other academic domains is a question we leave for the readers to think about.

The program of proofs through premises, derivation, and conclusion has clear limits. Not all inferences can be expressed as premises and conclusions: some of our experiences are ineffable (not expressible in words). Even when the premises are expressed in terms of the words, phrases, and sentences of a natural language, not all of their logical consequences are sanctioned by the rules of inference in a system of logic. There are also limits to the clarity and precision that a human language can achieve. In spite of these limits, however, this program remains a remarkable achievement for the search for reliable knowledge that the human species is capable of.

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